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Learning About Monetary Policy Rules

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Abstract. We study macroeconomic systems with forward-looking private sector agents and a monetary authority that is trying to control the economy through the use of a linear policy feedback rule. We use stability under recursive learning *a la* Evans and Honkapohja (2001) as a criterion for evaluating monetary policy rules in this context. We find that considering learning can alter the evaluation of alternative policy rules. *JEL Classification* E4, E5.

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1. INTRODUCTION

1.1. Overview. Monetary policy rules have been the subject of a good deal of recent research in the literature on monetary economics and monetary policy.[1] While some of this work has focussed on systems which abstract from or suppress private sector expectations, many of the more recent papers analyze systems where private sector expectations enter the model explicitly. Most of these models involve small, forward-looking representations of the macroeconomy, such as those found in Clarida, *et al.*, (1999), McCallum and Nelson (1999), and Woodford (1999). In many cases the small model is a log-linearized and simplified version of a larger model derived from optimizing behavior in a dynamic stochastic general equilibrium context.

When private sector expectations enter such models explicitly, recent research has emphasized the possibility that certain policy rules may be associated with indeterminacy of rational expectations equilibrium, and therefore might be viewed as undesirable. Some of the authors that discuss this issue include Bernanke and Woodford (1997), Carlstrom and Fuerst (1999, 2000), Christiano and Gust (1999), Clarida, *et al.*, (2000), Rotemberg and Woodford (1998, 1999), and Woodford (1999). In a typical analysis, the authors compute the rational expectations solutions of the system with a given monetary policy rule, and if the rule induces indeterminacy then it is viewed as undesirable. The idea is that if the monetary authorities actually followed such a rule, the system might be unexpectedly volatile as agents are unable to coordinate on a particular equilibrium among the many that exist.[2] In contrast, when equilibrium is determinate, it is normally assumed that the agents can coordinate on that equilibrium.

It is far from clear, however, exactly how or whether such coordination would arise. In order to complete such an argument, one needs to show the potential for agents to learn the equilibrium of the model being analyzed. In this paper, we take on this task.

[1] For a sample of the recent work, see the volume edited by Taylor (1999).

[2] Alternatively, the agents may be able to coordinate, but the risk exists that the equilibrium achieved may be one with undesirable properties, such as a large degree of volatility. See Woodford (1999, pp. 67-69).

We assume the agents of the model do not initially have rational expectations, and that they instead form forecasts by using recursive learning algorithms—such as recursive least squares—based on the data produced by the economy itself. Our methodology is that of Evans and Honkapohja (1999, 2001). We ask whether the agents in such a world can learn the fundamental or MSV equilibrium of the system under a range of possible Taylor-type monetary policy feedback rules. We use the criterion of *expectational stability* (*a.k.a. E-stability*) to calculate whether rational expectations equilibria are stable under real time recursive learning dynamics or not. The research of Marcet and Sargent (1989) and Evans and Honkapohja (1999, 2001) has shown that the expectational stability of rational expectations equilibrium governs local convergence of real time recursive learning algorithms in a wide variety of macroeconomic models.[3]

We think of learnability as a necessary additional criterion for evaluating alternative monetary policy feedback rules. In particular, in our view economists should only advocate policy rules which induce learnable rational expectations equilibria. Central banks adopting monetary policy rules that are not associated with learnable rational expectations equilibria, under the assumption that private sector agents will coordinate on the equilibrium they are targeting, are making an important mistake. Our analysis suggests that such policymakers will encounter difficulties, as the private sector agents instead fail to coordinate, and the macroeconomic system diverges away from the targeted equilibrium. Learnable equilibria, on the other hand, do not have such problems. This is because the agents can indeed coordinate on the equilibrium the policymakers are targeting, so that the learning dynamics tend toward, and eventually coincide with, the rational expectations dynamics. Learnable equilibria are therefore to be recommended.

1.2. Model environment. We consider monetary policy rules which have been suggested by various authors. All of these rules envision the central bank adjusting a short-term nominal interest rate in linear response to deviations of inflation from some target

[3] Accordingly, we use the terms "learnability," "expectational stability," "E-stability," and "stability in the learning dynamics" interchangably in this paper.

level and to deviations of real output from some target level. We take up four variants of such rules which we believe are representative of the literature: rules where the nominal interest rate set by the central bank responds to deviations of current values of inflation and output (we call this the *contemporaneous data* specification); rules where the interest rate reacts to lagged values of output and inflation deviations (*lagged data* specification); rules where the interest rate responds to future forecasts of inflation and output deviations (*forward looking* rules); and finally, rules which respond to current expectations of inflation and output deviations (*contemporaneous expectations*).

The novel contribution of this paper is to evaluate these policy rules based on the learnability criterion in a standard, small, forward-looking macroeconomic model which is currently the workhorse for the study of such rules. We analyze the stability of equilibria under learning dynamics, and we also provide conditions for unique equilibria. Conditions for unique equilibria may be found sporadically for some of these policy rules in the existing literature, and we put these results into a unifying framework. Thus, we are able to evaluate monetary policy rules based not only on whether they induce determinacy but also based on whether they induce learnability.

1.3. Main results. We find that monetary policy rules which react to current values of inflation and output deviations can easily induce determinate equilibria. Moreover, when equilibrium is determinate it is also learnable under this specification. However, contemporaneous data rules have often been criticized because they place unrealistic informational demands on the central bank, since precise information on current quarter values of inflation and output is usually not available to policymakers. One of our important findings is that rules which react to contemporaneous *expectations* of inflation and output deviations lead to exactly the same regions of determinate and learnable equilibria. Consequently, our results suggest that rules where the central bank responds to current expectations of inflation and output deviations are the most desirable in terms of generating both determinacy and learnability. Our reading of the policy rules literature is that such rules have not been given adequate attention and our results suggest more emphasis on them may prove fruitful.

We find that rules which respond to lagged values or to future forecasts of inflation and output deviations do not have the same desirable properties. Determinate rational expectations equilibria are not necessarily learnable under the lagged data specification. In addition, rules which respond to lagged data can easily fail to generate determinacy. Forward-looking rules can easily induce equilibrium indeterminacy (see also Bernanke and Woodford (1997)). We find that determinate equilibria are always learnable for forward-looking rules, but when equilibrium is indeterminate, those equilibria which correspond to the minimum state variable (MSV) solutions may also be learnable. We do not examine the learnability of sunspot equilibria, which may exist when the equilibrium is indeterminate, in this paper.

Taylor (1999a) recommends a "leaning against the wind" policy rule which calls for nominal interest rates which are adjusted positively, and more than one-for-one, in response to inflation above target, and positively to levels of production above target. We call this the *Taylor principle* following Woodford (2000, 2001). Taylor's intuition is that under such a rule, a rise in inflation brings about an increase in the real interest rate which reduces demand and inflationary pressures, bringing the economy back towards the targeted equilibrium. On the other hand, a policy rule which does not obey the Taylor principle brings about a decrease in the real interest rate which adds to inflationary pressures, pushing the economy away from the targeted equilibrium.

In this paper we support Taylor's intuition based on the criterion of learnability. In fact, we find that the Taylor principle completely characterizes learnability. If agents do not have rational expectations of inflation and output and instead start with some subjective expectations of these variables, learning recursively using some version of least squares, then a "leaning against the wind" policy on the part of the central bank does indeed push the economy towards the rational expectations equilibrium (REE) across all the specifications of policy rules we consider.

Finally, we find that across all the information structures we consider, policy rules which respond relatively aggressively to inflation with little or no reaction to the output gap (or output gap forecasts) generally induce both determinate and learnable rational expectations equilibria. To the extent that both determinacy and learnability are desirable criteria, central banks may want to consider adopting such rules.

1.4. Recent related literature. In an analysis complimentary to ours, Evans and Honkapohja (2000) consider the learnability of equilibria induced by optimal monetary policy rules in a structural model like the one used in this paper. By *optimal*, Evans and Honkapohja mean a policy rule derived from minimization of a loss function for the monetary authority, given the structure of the economy. A rule derived in such a way may or may not generate either determinacy or learnability of equilibrium, and Evans and Honkapohja investigate both of these properties. One important finding is that if the central bank assumes rational expectations on the part of private agents, then the equilibria induced are always unstable in the learning dynamics. On the other hand, if optimal policy is conditioned directly on the observed (subjective) private sector expectations, then the REE becomes stable under learning dynamics.

In contrast to Evans and Honkapohja (2000), we study simple policy rules which are of the form recommended in Taylor's (1993) widely-cited work. We locate the set of rules in this class which are associated with both determinacy and learnability. A practitioner wishing to find an optimal policy rule in this set could then postulate an objective criterion for the central bank and use it to locate the best rule. This is essentially the same process that Rotemberg and Woodford (1998, 1999) and other authors have used to analyze these types of rules. The advantage of remaining in the class of Taylor-type rules hinges on the alleged robustness of these rules across models, as discussed at length in the Taylor (1999) volume. For this reason we think it is interesting to consider learnability under either optimal rules or Taylor-type rules.

McCallum (1999) and Taylor (1999b) have argued that it is important to check the robustness of policy rules in different monetary models since in general there is little agreement among economists about the appropriateness of any particular model. In this respect, Carlstrom and Fuerst (1999, 2000) show that the equilibrium determinacy properties in models like the one we analyze are sensitive to certain key assumptions. These assumptions include which money balances, in terms of timing, enter the utility function, and also the nature of the sticky price assumption along off-equilibrium paths. They conclude that under their alternative assumptions, in setting the nominal interest rate, central banks should react aggressively to lagged inflation in order to preserve determinacy of equilibrium. Our model also provides support for policy rules which react aggressively to lagged inflation (and mildly to output). We also advocate policy rules based on contemporaneous forecasts. Since these forecasts are in effect based on past data in our systems under learning, we support the intuition that a central bank should look backward, as in the models analyzed by Carlstrom and Fuerst. We also think it would be interesting to carry out our learning analysis in the classes of models analyzed by Carlstrom and Fuerst.

1.5. Organization. In the next section we present the model we will analyze throughout the paper. We also discuss the types of linear policy feedback rules we will use to organize our analysis, and a calibrated case which we will employ. In the subsequent sections, we present results on determinacy of equilibrium, and then on learnability of equilibrium, for each of four different classes of policy rules. We conclude with a summary of our findings.

2. The environment

2.1. A baseline model. We study a simple and small forward-looking macroeconomic model analyzed by Woodford (1999). Woodford (1999) derived his model from a more elaborate, optimizing framework with sticky prices studied by Rotemberg and Woodford (1998, 1999), and intended it to be a parsimonious description of the U.S. economy, with mechanisms that would remain prominent in nearly any model with complete microfoundations. We write Woodford's (1999, p. 16) system as

$$z_t = z_{t+1}^e - \sigma^{-1}\left(r_t - r_t^n - \pi_{t+1}^e\right) \tag{1}$$

$$\pi_t = \kappa z_t + \beta \pi_{t+1}^e \tag{2}$$

where z_t is the output gap, π_t is the inflation rate, and r_t is the nominal interest rate. Each variable is expressed as a percentage deviation from its long run level, and in particular, r_t is the deviation of the nominal interest rate from the steady state value that would obtain with zero inflation and steady state output growth. A superscript "e" represents a (possibly nonrational) expectation. We use this notation for expectations so that we can be flexible in describing our systems under both rational expectations and learning, along with the accompanying informational assumptions in each case. The parameters σ, relating to the elasticity of intertemporal substitution of the representative household, κ, relating to the degree of price stickiness, and β, the household's discount factor, are viewed as structural, arising from the analysis of the larger dynamic stochastic general equilibrium model, and from the subsequent approximations to that model that produce these equations. We assume κ and σ are positive on economic grounds, and that $0 < \beta < 1$. The "natural rate of interest" r_t^n is an exogenous stochastic term that follows the process

$$r_t^n = \rho r_{t-1}^n + \epsilon_t \tag{3}$$

where ϵ_t is iid noise with variance σ_ϵ^2, and $0 \le \rho < 1$ is a serial correlation parameter.

We supplement equations (1), (2), and (3) with a policy rule, which represents the behavior of the monetary authority. Our baseline specification is to use

$$r_t = \varphi_\pi \pi_t + \varphi_z z_t \tag{4}$$

so that r_t is set by the monetary authority in response to inflation and the size of the output gap.[4] We assume throughout the paper that φ_π and φ_z are non-negative, with at least one strictly positive.

For notational convenience we have written the policy rule (4), as well as the ones in the next section, without a constant term. Woodford (1999, pp. 17-18) has argued that in general, a nonzero term $r^\star$ should be added to the right hand side of equation (4) to account for the idea that in this model, the policy authorities may well wish to stabilize nominal interest rates about a value different from the steady state value consistent with zero inflation and steady state output growth. This would occur either because

[4]Rotemberg and Woodford (1999) respond to the Lucas critique by arguing that the equations describing their economy have coefficients which are not dependent on the parameters in the monetary authority's policy rule; accordingly they study a number of possible policy rules in their paper. We follow their procedure in this paper.

the policy authority wishes to counteract the distortions emphasized by Friedman (1969), or because the authorities wish to account for the possibility of encountering the lower bound on nominal interest rates. The constant term makes little difference for the issues we discuss (and hence we suppress it), except for one point: When we analyze learning in the remainder of the paper, we will allow agents to take account of a possible constant term in their learning rule, because this is the more general specification. In the special case where there is actually no constant in the economic model and agents also specify no constant in their learning rule, the E-stability conditions we report will be somewhat altered. We regard this as a special case, and we do not conduct an extensive analysis of it.

2.2. Alternative specifications for setting interest rates. In the model given by equations (1)-(4), only the private sector forms expectations about future values of endogenous variables. The policymakers, whose behavior is embodied in equation (4), only react to information which is observed at time t. McCallum (1999) has argued that such reaction functions are unrealistic, since actual policymakers do not have complete information on variables such as output and inflation in the quarter they must make a decision. In reaction to this critique, we will call the specification embodied in equation (4) our *contemporaneous data* specification, and we will consider other possibilities below.[5]

One alternative is to follow one of McCallum's suggestions and posit that the monetary authorities must react to last quarter's observations on inflation and the output gap, which could possibly be viewed as closer to the reality of central bank practice. This leads to

[5] There is an additional problem with the contemporaneous data specification. When private sector expectations are formed using information dated $t-1$ and earlier, a tension is introduced, because the monetary authority is reacting to time t information on inflation and the output gap. Thus the central bank has "superior information" in this specification. In our other specifications, this tension is absent, as the private sector and the central bank use the same information, either for forming expectations or setting the interest rate instrument, or both.

our *lagged data* specification for our interest rate equation, in which (4) is replaced with

$$r_t = \varphi_\pi \pi_{t-1} + \varphi_z z_{t-1}. \tag{5}$$

Another method of coping with McCallum's criticism is to assume that the authorities set their interest rate instrument in response to their *forecasts* of output and inflation deviations, formed using the information available as of either time $t-1$ or time t, so that the policy rule itself is forward-looking. Bernanke and Woodford (1997), among others, have analyzed forward-looking rules. We consider simple versions of such rules, ones in which the monetary authority looks just one quarter ahead when setting its interest rate instrument. This yields a specification, which we call the *forward expectations* model, in which the interest rate equation (4) is replaced with

$$r_t = \varphi_\pi \pi^e_{t+1} + \varphi_z z^e_{t+1}. \tag{6}$$

There are several ways to interpret this equation. When there is learning in the model, it may be *two-sided*, as both policymakers and private sector agents form (identical) expectations of the future. We impute identical learning algorithms to each when we introduce learning. Alternatively, following Bernanke and Woodford (1997), it may be that the central bank simply reacts to the predictions of private sector forecasters, so that in this interpretation it is only the private sector which is learning.

Finally, another way to cope with McCallum's criticism is to assume that the authorities set their interest rate instrument in response to their current *expectations* (as opposed to using current data), formed using the information available as of time $t-1$, of the current period output gap and inflation. This also might be viewed as close to the actual practice of central banks. The interpretations given above for the case when the central bank targets future forecasts of inflation and output carry over to this case. Thus we consider *contemporaneous expectations* versions of our systems where the policy feedback rule (4) is replaced with

$$r_t = \varphi_\pi \pi^e_t + \varphi_z z^e_t. \tag{7}$$

2.3. Methodology. We begin our analysis of each system by providing conditions for a determinate equilibrium to exist. We use standard methodology for this purpose, whereby the system is required to have the correct number of eigenvalues inside the unit circle given the number of free and predetermined variables in the system.[6]

We adapt methods developed by Evans and Honkapohja (1999, 2001) to understand how learning affects these systems. We assume the agents in the model no longer have rational expectations at the outset. Instead, we replace expected values with adaptive rules, in which the agents form expectations using the data generated by the system in which they operate. We imagine that the agents use versions of recursive least squares updating, and we calculate the conditions for expectational stability (E-stability). Evans and Honkapohja (2001) have shown that expectational stability, a notional time concept, corresponds to stability under real-time adaptive learning under quite general conditions. In particular, under E-stability recursive least squares learning is locally convergent to the rational expectations equilibrium (REE). Moreover, under weak assumptions, it can be shown that if a rational expectations equilibrium is not E-stable, then the probability of convergence of the recursive least squares algorithm to the rational expectations equilibrium is zero.

We now define precisely the concept of E-stability. Following Evans and Honkapohja (2001), consider a general class of models[7]

$$y_t = \alpha + BE_t y_{t+1} + \delta y_{t-1} + \varkappa w_t \tag{8}$$

$$w_t = \phi w_{t-1} + e_t \tag{9}$$

where y_t is an $n \times 1$ vector of endogenous variables, α is an $n \times 1$ vector of constants, B, δ, $\varkappa$, and ϕ are $n \times n$ matrices of coefficients, and w_t is an $n \times 1$ vector of exogenous variables which is assumed to follow a stationary VAR, so that e_t is an $n \times 1$ vector of

[6] For discussions of this methodology in contexts like ours see Blanchard and Kahn (1980), McCallum (1983), and Farmer (1991, 1999).

[7] The class of models discussed in the previous section are special cases of this general class. A similar analysis can also be carried out when expectations are dated time $t-1$.

white noise terms. Following McCallum (1983), we focus on the MSV (minimal state variable) solutions which are of the form

$$y_t = a + by_{t-1} + cw_t \tag{10}$$

where a, b, and c are conformable and are to be calculated by the method of undetermined coefficients. Use (10) to compute the corresponding expectation, $E_t y_{t+1} = a + by_t + c\phi w_t$, and the MSV solutions consequently satisfy

$$(I - Bb - B)a = \alpha, \tag{11}$$

$$Bb^2 - b + \delta = 0, \tag{12}$$

and

$$(I - Bb)c - Bc\phi = \varkappa. \tag{13}$$

For E-stability we regard equation (10) as the *perceived law of motion* (PLM) of the agents and using $E_t y_{t+1} = a + by_t + c\phi w_t$, one obtains the *actual law of motion* (ALM) of y_t as

$$y_t = (I - Bb)^{-1}[\alpha + Ba + \delta y_{t-1} + (Bc\phi + \varkappa)w_t]. \tag{14}$$

Thus the mapping from the PLM to the ALM takes the form

$$T(a, b, c) = \left((I - Bb)^{-1}(\alpha + \beta a)\right), (I - Bb)^{-1}\delta, (I - Bb)^{-1}(Bc\phi + \varkappa)). \tag{15}$$

Expectational stability is determined by the following matrix differential equation

$$\frac{d}{d\tau}(a, b, c) = T(a, b, c) - (a, b, c). \tag{16}$$

The fixed points of equation (16) give us the MSV solution. We say that a particular MSV solution $(\bar{a}, \bar{b}, \bar{c})$ is E-stable if the MSV fixed point of the differential equation (16) is locally asymptotically stable at that point. The conditions for E-stability of the MSV solution $(\bar{a}, \bar{b}, \bar{c})$ are given in Proposition 10.3 of Evans and Honkapohja (2001).[8]

[8] For an interpretation of equation (16) as a stylized learning process, see Evans and Honkapohja (2001, Chapter 2).

Under real time learning, the PLM is time dependent and takes the form

$$y_t = a_{t-1} + b_{t-1}y_{t-1} + c_{t-1}w_t \tag{17}$$

where the coefficients a_t, b_t, c_t are updated by running recursive least squares on actual data, $x_t' = (1, y_{t-1}', w_t)$. This generates a corresponding ALM for y_t (which is also obviously time dependent). However, as shown in Proposition 10.4 of Evans and Honkapohja (2001), the E-stability conditions derived from equation (16) actually govern stability under such adaptive learning. This is the reason why we focus on E-stability conditions throughout the paper.

In the learning literature, an important issue is the so-called "dating of expectations." That is, when an expectation term enters the model, there is a question of what information the agent is able to incorporate when forming expectations. Expectational stability conditions, in general, are influenced by the exact dating of expectations. The convention in the learning literature is to assume all expectations are formed at time t using information available as of time $t-1$. Evans and Honkapohja (1999) comment that this assumption "... seems more natural in a learning environment." If one assumes instead that time t observations are in the information set, then a simultaneity problem is introduced, where the system is determining time t variables at the same time that agents are using time t variables to form expectations. The problem can usually be handled at the cost of additional complexity. In this paper we generally work out results for both t and $t-1$ dating of expectations and where appropriate report on any differences we find.

2.4. Parameters. We have analytical results in most cases, but we illustrate our findings using a calibrated case. Woodford (1999) calibrated the parameters $\sigma = .157$ and $\kappa = .024$ based on econometric estimates from a larger model contained in Rotemberg and Woodford (1998, 1999); we use these as our baseline values. The value of the discount factor β is set throughout to .99. Calibrations of the policy rules correspond to values for the parameters φ_π and φ_z, and we examine $0 \leq \varphi_\pi \leq 10$, and $0 \leq \varphi_z \leq 4$. For the stochastic process describing the natural rate of interest, we use $\rho = .35$, the serial correlation suggested by Woodford (1999). We also discuss the robustness of our results to an alternative calibration suggested by Clarida, *et al.* (2000). In that calibration, we increase the values of two parameters: σ from .157 to 1 and κ from .024 to .3.

3. Policy rules and the equilibria they induce

3.1. Contemporaneous data in the policy rule.

Determinacy. In this subsection, we consider the contemporaneous data version of the model represented by equations (1)-(4). We substitute the policy rule (4) into (1), and put our system involving the two endogenous variables z_t and π_t (given by equations (1) and (2)) in the following form[9]

$$y_t = \alpha + By_{t+1}^e + \varkappa r_t^n \tag{18}$$

where $y_t = [z_t, \pi_t]'$, $\alpha = 0$, and

$$B = \frac{1}{\sigma + \varphi_z + \kappa\varphi_\pi} \begin{bmatrix} \sigma & 1 - \beta\varphi_\pi \\ \kappa\sigma & \kappa + \beta(\sigma + \varphi_z) \end{bmatrix}, \tag{19}$$

where the form of $\varkappa$ is omitted since it is not needed in what follows. Since both z_t and π_t in the system (18) are free, we need both of the eigenvalues of B to be inside the unit circle for determinacy; otherwise the equilibrium will be indeterminate. We provide a characterization of the necessary and sufficient condition for determinacy in the following proposition.

Proposition 1. *Let* $\kappa(\varphi_\pi - 1) + (1-\beta)\varphi_z \neq 0$.[10] *Under contemporaneous data interest rate rules the necessary and sufficient condition for a rational expectations equilibrium to be unique is that*

$$\kappa(\varphi_\pi - 1) + (1-\beta)\varphi_z > 0. \tag{20}$$

Proof. See Appendix A. ■

Condition (20) may be given an economic interpretation and this is discussed in the next section.

[9] One obtains a similar system whether expectations are dated time t or time $t-1$.

[10] This condition rules out non-generic cases which we will generally ignore in what follows.

Learning. The results for learning are identical whether we assume $t-1$ or t dating of expectations in (18). For convenience, we assume t dating of expectations below. The MSV solution in this case takes the simple form $y_t = \bar{a} + \bar{c} r_t^n$ with $\bar{a} = 0$ and $\bar{c} = (I - \rho B)^{-1} \varkappa$ (where I denotes a conformable identity matrix throughout the paper). For the study of learning, we endow agents with a perceived law of motion (PLM)

$$y_t = a + c r_t^n \tag{21}$$

which corresponds to the MSV solution. As we mentioned in the discussion of the policy rule (4), the more general specification of the PLM involves a constant term for this model. Using the PLM, we compute $E_t y_{t+1} = a + c\rho r_t^n$, assuming the time t information set to be $(1, r_t^n)'$, and substituting this into equation (18), we obtain the actual law of motion (ALM) which is followed by y_t as

$$y_t = Ba + (Bc\rho + \varkappa) r_t^n. \tag{22}$$

Using (21) and (22), we can define a map, T, from the PLM to the ALM as

$$T(a, c) = (Ba, Bc\rho + \varkappa). \tag{23}$$

Expectational stability is then determined by the matrix differential equation as discussed earlier.

We now compute the necessary and sufficient condition for a MSV solution to equation (18) to be E-stable using the framework spelled out above. This yields an important baseline result, which is that the condition that guarantees E-stability turns out to be identical to the condition which guarantees uniqueness of rational expectations equilibrium.

Proposition 2. *Let $\kappa(\varphi_\pi - 1) + (1 - \beta)\varphi_z \neq 0$. Suppose the time t information set is $(1, r_t^n)'$. Under contemporaneous data interest rate rules, the necessary and sufficient condition for an MSV solution $(0, \bar{c})$ of (18) to be E-stable is that*[11]

$$\kappa(\varphi_\pi - 1) + (1 - \beta)\varphi_z > 0. \tag{24}$$

[11] It is possible to show that the MSV solution is learnable even when agents allow for a lag in the endogenous variables in their PLM, so that this result is robust at least to some overparametrizations in the PLM of agents (see Evans and Honkapohja (2001)).

Proof. See Appendix B. ■

The key condition in Propositions 1 and 2, inequality (24), can be rewritten as

$$\varphi_\pi + \frac{(1-\beta)}{\kappa}\varphi_z > 1. \tag{25}$$

The price adjustment equation (2) implies that each percentage point of permanently higher inflation results in a permanently higher output gap of $(1-\beta)/\kappa$ percentage points. The left hand side of (25) then shows the long-run increase in the nominal interest rate prescribed by the interest rule (4) for each unit permanent increase in the inflation rate. Conditon (24), therefore, corresponds to the *Taylor principle* as discussed by Woodford (2000, 2001): Nominal interest rates rise by more than the increase in the inflation rate in the long-run. The intuition for this result is quite clear. A deviation of private sector expected inflation from the rational expectations (RE) value leads to an increase in the real interest rate when the Taylor principle is satisfied. This reduces the output gap through equation (1) which in turn reduces inflation through equation (2). Such a policy, therefore, succeeds in guiding initially nonrational private sector expectations towards the RE value. On the other hand, if the policy rule does not satisfy this principle, a deviation of private sector expected inflation from the RE value leads to a decrease in the real interest rate which increases the output gap through (1) and increases inflation through (2). Over time, this leads to upward revisions of both expected inflation and expected output gap. The interest rate rule is unable to offset this tendency and the economy moves further away from the rational expectations equilibrium. Obviously, $\varphi_\pi > 1$ is sufficient for the Taylor principle to be satisfied. But even for values of $\varphi_\pi < 1$, the policy authority can compensate for a relatively low value of φ_π by choosing a sufficiently large value of φ_z, in such a way as to still satisfy condition (25). One of our important findings is that the Taylor principle seems to characterize learnability for all classes of policy rules we study.

Propositions 1 and 2 in conjunction show that under contemporaneous data policy rules, the set of parameter values consistent with determinate equilibria are *exactly the same* as the set consistent with expectational stability. Since all determinate REE are E-stable, one could view this as justifying the focus on determinate equilibria in previous studies of policy rules, for the case in which the policy rule reacts to contemporaneous data.

[FIGURE 1 ABOUT HERE].

Figure 1 plots the region of determinacy and expectational stability of the MSV solution as a function of φ_π and φ_z, when all other parameters are set at baseline values. Much of the parameter space is associated with a determinate REE. Our result under learning shows that this entire region is also associated with expectational stability. In the indeterminate region of the parameter space, near the origin in the Figure, equilibria corresponding to the MSV solution are always expectationally unstable when agents have a PLM corresponding to the relevant MSV solution. Note that the negatively sloped straight line in the figure corresponds to the Taylor principle; points to the right of this line are learnable (and determinate, in this case) whereas points to the left are not learnable (and indeterminate, in this case). This line has a vertical intercept corresponding to $\varphi_z = \kappa/(1-\beta)$ and a horizontal intercept corresponding to $\varphi_\pi = 1$. Consequently, the line pivots upwards as κ increases with the horizontal intercept staying unchanged. As a result, for the Clarida, *et al.*, (2000) calibration, which uses a higher κ $(= .3)$, the vertical intercept becomes larger $(= 30)$. Our analytical results, therefore, show that higher values of κ increase the size of the indeterminate region near the origin (as long as $\varphi_\pi < 1$) whereas it leaves the determinate region for $\varphi_\pi > 1$ unaffected.

As we have noted above, the economic model as we have written it in equations (1) through (4) has no constant terms, but the perceived law of motion of the agents under learning (21) does have a constant term. If we included explicit constant terms in the economic model (1) through (4) and proceeded with the same analysis, the E-stability conditions would be identical to the ones we report in this section and below. If, however, the model (1) through (4) is literally interpreted as not having any constant terms—and Woodford (1999) suggests that this is a difficult case to make—then the appropriate perceived law of motion for the agents under learning would also not include a constant term. In this special case, the condition for E-stability would change somewhat. In particular, the condition would depend on ρ, the degree of serial correlation of the shock, and if $\varphi_z = 0$ then the condition would reduce to $\varphi_\pi > \rho$ instead of $\varphi_\pi > 1$. Thus the condition for E-stability would be less stringent. However, we regard this situation as a special case and we do not pursue it further in the remainder of the paper.

3.2. Lagged data in the policy rule.

Determinacy. The case of a policy rule with contemporaneous data is probably the least realistic in terms of what policymakers actually know when decisions about interest rates are made. In this subsection, we consider rules with lagged data, so that policymakers are, in the current quarter, reacting to information about the previous quarter's output gap and inflation rate. The policy rule is, therefore, given by (5) and our complete system is given by (1), (2), (3) and (5).

For the analysis of uniqueness, we move equation (5) one time period forward and rewrite the system of equations (1), (2), and (5) as

$$\begin{bmatrix} 1 & 0 & \sigma^{-1} \\ -\kappa & 1 & 0 \\ \varphi_z & \varphi_\pi & 0 \end{bmatrix} \begin{bmatrix} z_t \\ \pi_t \\ r_t \end{bmatrix} = \begin{bmatrix} 1 & \sigma^{-1} & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_{t+1}^e \\ \pi_{t+1}^e \\ r_{t+1} \end{bmatrix} + \begin{bmatrix} \sigma^{-1} \\ 0 \\ 0 \end{bmatrix} r_t^n. \tag{26}$$

Pre-multiplying the matrix associated with the expectational variables with the inverse of the left hand matrix gives

$$B = \frac{1}{(\varphi_z + \kappa\varphi_\pi)} \begin{bmatrix} 0 & -\beta\varphi_\pi & 1 \\ 0 & \beta\varphi_z & \kappa \\ \sigma(\varphi_z + \kappa\varphi_\pi) & \varphi_z + (\kappa + \beta\sigma)\varphi_\pi & -\sigma \end{bmatrix} \tag{27}$$

which is the matrix relevant for uniqueness. We now have two free endogenous variables, z_t and π_t, and one predetermined endogenous variable, r_t. Consequently, we need exactly two of the eigenvalues of B to be inside the unit circle for uniqueness.

In order to have a unique equilibrium, we must rule out aggressive response to either inflation or output as is shown in the proposition below.[12]

Proposition 3. *Under lagged data interest rate rules a set of sufficient conditions for unique equilibria are*

$$\kappa(\varphi_\pi - 1) + (1 - \beta)\varphi_z > 0, \tag{28}$$

[12] In this case it can be shown that a sufficiently aggressive response to inflation and output will necessarily lead to local *explosiveness*, that is, paths that are locally diverging away from the steady state.

and

$$\kappa(\varphi_\pi - 1) + (1+\beta)\,\varphi_z < 2\sigma(1+\beta). \tag{29}$$

Proof. See Appendix C. ■

Proposition 3 shows that rules characterized by $\varphi_\pi > 1$ and a small response to output can lead to unique equilibria. Proposition 2 established that determinate equilibria are the only learnable ones, for the case of contemporaneous data in the policy rule. This baseline result concerning learnability does not carry over to the case of lagged data interest rate rules, however, as we now show.

Learning. For the analysis of learning, we substitute equation (5) into equation (1) and reduce the system to two equations involving the endogenous variables z_t and π_t. Defining $y_t = [z_t, \pi_t]'$, this system can be written as

$$y_t = \beta_1 y^e_{t+1} + \delta y_{t-1} + \varkappa r^n_t \tag{30}$$

with

$$\beta_1 = \begin{bmatrix} 1 & \sigma^{-1} \\ \kappa & \kappa\sigma^{-1} + \beta \end{bmatrix}, \tag{31}$$

$$\delta = \begin{bmatrix} -\varphi_z\sigma^{-1} & -\varphi_\pi\sigma^{-1} \\ -\kappa\varphi_z\sigma^{-1} & -\kappa\varphi_\pi\sigma^{-1} \end{bmatrix}, \tag{32}$$

and the form of $\varkappa$ is omitted since it is not needed. In this formulation, we assume that expectations of inflation and output of the private sector are formed at time $t-1$.[13] The MSV solution of (30) takes the form

$$y_t = \bar{a} + \bar{b} y_{t-1} + \bar{c} r^n_{t-1} + \varkappa\epsilon_t \tag{33}$$

with $\bar{a} = 0$, $\bar{c} = \rho(I - \beta_1\bar{b} - \rho\beta_1)^{-1}\varkappa$, and

$$\bar{b} - \beta_1\bar{b}^2 - \delta = 0. \tag{34}$$

[13] Since the central bank is using last quarter's values of inflation and output in setting the current interest rate, assuming that the private sector has access to current quarter values of inflation and output in forming its expectations is tantamount to assuming that the public has superior information. Assuming $t-1$ dating of expectations removes this tension and puts the monetary authority and the public in a symmetric position.

Because equation (34) is a matrix quadratic, there are potentially multiple solutions for $\bar{b}$. The determinate case corresponds to the situation when there is a unique solution for $\bar{b}$ with both its eigenvalues inside the unit circle. For the analysis of learning, we assume that agents have a PLM of the form

$$y_t = a + by_{t-1} + cr_{t-1}^n + \varkappa\epsilon_t \tag{35}$$

corresponding to the MSV solution which leads to an ALM of the form

$$y_t = (\beta_1 + \beta_1 b)a + (\beta_1 b^2 + \delta)y_{t-1} + (\beta_1 bc + \rho\beta_1 c + \rho\varkappa)r_{t-1}^n + \varkappa\epsilon_t. \tag{36}$$

The mapping from the PLM to the ALM can then be computed, and we can apply the results of Evans and Honkapohja (2001). We require the eigenvalues of all of the three matrices $\bar{b}' \otimes \beta_1 + I \otimes \beta_1\bar{b}$, $\rho\beta_1 + \beta_1\bar{b}$, and $\beta_1 + \beta_1\bar{b}$ to have real parts less than one for E-stability whereas E-instability obtains if any eigenvalue of these matrices has a real part more than one.

[FIGURE 2 ABOUT HERE.]

We did not obtain analytical results in this case. However, we illustrate our findings in Figure 2, which depicts determinacy and learnability, when all parameters other than φ_π and φ_z are set at baseline values. We first note that only a subset of the parameter space that is consistent with determinacy is also consistent with learnability in the lagged data case. Determinate but E-unstable equilibria exist for values of $\varphi_\pi < 1$ and relatively high values of φ_z. On the other hand, the determinate equilibria for values of $\varphi_\pi > 1$ and small values of φ_z are E-stable. In the indeterminate region of the parameter space, which occurs for values of $\varphi_\pi < 1$ and relatively small values of φ_z, we find that there are two stationary solutions which take the form of the MSV solution (33). However, both of these stationary solutions always turn out to be E-unstable. For the alternative calibration suggested by Clarida, *et al.*, (2000), we find that the boundary between the determinate and learnable region and the explosive region shifts upward in the parameter space, i.e., roughly speaking, we obtain determinacy and learnability for a larger portion of the parameter space. Qualitatively, however, the regions are similar: there continue to exist determinate E-unstable equilibria for values of $\varphi_\pi < 1$ and relatively high values of

φ_z, determinate E-stable equilibria for values of $\varphi_\pi > 1$ and relatively small values of φ_z, and indeterminate equilibria for values of $\varphi_\pi < 1$ and relatively small values of φ_z. In the latter case, both of the stationary MSV solutions continue to be E-unstable.

We checked that all (stationary) MSV solutions which satisfy the Taylor principle are E-stable while all solutions which do not satisfy the Taylor principle are E-unstable (whether in the determinate or indeterminate region of the parameter space). The intimate connection we observe between E-stability and policy rules which conform to the Taylor principle leads us to conjecture that condition (24) still characterizes learnability in the lagged data case. On the other hand, determinacy is not characterized by the Taylor principle in this case.

3.3. Forward expectations in the policy rule.

Determinacy. With forward expectations, the policy rule is given by (6), and the complete model by (1), (2), (3), and (6). We can again reduce the system of equations (1), (2), and (6) to two equations involving the endogenous variables (z_t, π_t) by substituting equation (6) into equation (1). The reduced system is then given by

$$y_t = \alpha + By^e_{t+1} + \varkappa r^n_t \tag{37}$$

where $y_t = [z_t, \pi_t]'$, $\alpha = 0$, $\varkappa = [\sigma^{-1}, \kappa\sigma^{-1}]'$, and B is as defined by

$$B = \begin{bmatrix} 1 - \sigma^{-1}\varphi_z & \sigma^{-1}(1 - \varphi_\pi) \\ \kappa(1 - \sigma^{-1}\varphi_z) & \beta + \kappa\sigma^{-1}(1 - \varphi_\pi) \end{bmatrix}. \tag{38}$$

Since the variables z_t and π_t are free, we need both the eigenvalues of B to be inside the unit circle for uniqueness. In this case we are again able to provide a characterization of the necessary and sufficient conditions for determinacy. This is given in the following proposition.

Proposition 4. *Under interest rate rules with forward expectations the necessary and sufficient conditions for a rational expectations equilibrium to be unique are that*

$$\varphi_z < \sigma(1 + \beta^{-1}), \tag{39}$$

$$\kappa(\varphi_\pi - 1) + (1 + \beta)\varphi_z < 2\sigma(1 + \beta), \tag{40}$$

and

$$\kappa(\varphi_\pi - 1) + (1 - \beta)\varphi_z > 0. \tag{41}$$

Proof. See Appendix D. ■

We first note that, unlike the other specifications, values assigned to φ_z are of primary importance for determining uniqueness. In particular, an aggressive response to output leads to indeterminacy, quite independently of φ_π. If the response to output is modest, rules with $\varphi_\pi > 1$ can lead to determinate equilibria; however, too large a value of φ_π again leads to indeterminacy.

Learning. The analysis of E-stability here is akin to the case of rules with contemporaneous data and the results are identical for $t-1$ and t dating of expectations. For t-dating of expectations, the MSV solution takes the form $y_t = \bar{a} + \bar{c}r_t^n$ with $\bar{a} = 0$, and $\bar{c} = (I - \rho B)^{-1}\varkappa$. The PLM of agents again takes the form of equation (21) and the remainder of the analysis proceeds as in Section 3.1. We obtain a complete characterization for E-stability of the MSV solution in the following proposition.

Proposition 5. *Suppose the time t information set is $(1, r_t^n)'$. Under interest rate rules with forward expectations, the necessary and sufficient condition for an MSV solution $(0, \bar{c})$ of (37) to be E-stable is that*

$$\kappa(\varphi_\pi - 1) + (1 - \beta)\varphi_z > 0 \tag{42}$$

Proof. See Appendix E. ■

Propositions 4 and 5 in conjunction show that under policy rules with forward expectations, if an MSV solution is unique, then it must be expectationally stable. The condition is again that the Taylor principle applies as we showed in earlier cases. In this case, however, the converse does not hold, as satisfaction of the expectational stability condition does not imply satisfaction of the determinacy conditions. Consequently, when equilibrium is (potentially) indeterminate, the system may still converge to equilibria which correspond to the MSV solution.[14]

[14] We again stress that we have not undertaken the analysis of sunspot solutions in this paper. When such a solution exists, it remains an open question whether agents may be able to learn it in this context.

[FIGURE 3 ABOUT HERE].

Figure 3 illustrates the intersections of the regions of determinacy and learnability of the MSV solution at baseline parameter values. Determinate equilibria are always expectationally stable. For baseline values, this occurs (roughly) for values of φ_π between 1 and 27 and values of φ_z less than .32. On the other hand, in the indeterminate region of the parameter space, equilibria corresponding to the MSV solution may or may not be learnable when agents have a PLM corresponding to the relevant MSV solution. With the alternative calibration of Clarida, *et al.*, (2000), determinacy and learnability obtain (roughly) for values of φ_π between 1 and 13 and values of φ_z less than 2, i.e., for larger values of φ_z and smaller values of φ_π than in the baseline case. We also note that for given κ, higher values of σ enhance determinacy when $\varphi_\pi > 1$ since this relaxes the right hand constraints of equations (39) and (40).

3.4. Contemporaneous expectations in the policy rule.

Determinacy. With contemporaneous expectations, the policy rule is given by (7). In this case we assume $t-1$ dating of expectations for the central bank and the private sector so as to put them in a symmetric position as far as their information is concerned. Our complete model is given by (1), (2), (3), and (7). We can reduce our system of equations (1), (2), and (7) to two equations by substituting equation (7) into equation (1) and write this system in the form

$$y_t = B_0 y_t^e + B_1 y_{t+1}^e + \varkappa r_t^n \tag{43}$$

where $y_t = [z_t, \pi_t]'$, $\varkappa = [\sigma^{-1}, \kappa\sigma^{-1}]'$,

$$B_0 = \begin{bmatrix} -\varphi_z \sigma^{-1} & -\varphi_\pi \sigma^{-1} \\ -\kappa\varphi_z\sigma^{-1} & -\kappa\varphi_\pi\sigma^{-1} \end{bmatrix}, \tag{44}$$

and

$$B_1 = \begin{bmatrix} 1 & \sigma^{-1} \\ \kappa & \beta + \kappa\sigma^{-1} \end{bmatrix}. \tag{45}$$

The situation with contemporaneous expectations is similar to the situation with contemporaneous data at least as far as determinacy is concerned. In particular, the necessary and sufficient condition for a unique REE is given by condition (20) of Proposition 1 (and as portrayed in Figure 1 for baseline parameter values). The easiest way to see this is by replacing the expectations with the actual values (which is required for determining uniqueness following Farmer (1999) or Evans and Honkapohja (2001)) and observing that this yields the matrix, $(I - B_0)^{-1}B_1$, for determining uniqueness which is exactly the same as the one obtained for contemporaneous data, namely, the matrix given in (19). Furthermore, since the variables z_t and π_t are free, we need both eigenvalues of (19) to be inside the unit circle for uniqueness, as in the case of contemporaneous data.

Learning. Given our model of the form of equation (43), the MSV solution takes the form $y_t = \bar{a} + \bar{c}r^n_{t-1} + \varkappa\epsilon_t$ with $\bar{a} = 0$ and $\bar{c} = \rho(I - B_0 - \rho B_1)^{-1}\varkappa$. We assume that agents have the PLM

$$y_t = a + cr^n_{t-1} + \varkappa\epsilon_t \tag{46}$$

from which we compute the expectations $E_{t-1}y_t = a + cr^n_{t-1}$ and $E_{t-1}y_{t+1} = a + cE_{t-1}r^n_t = a + c\rho r^n_{t-1}$. Substituting this into our model (43) yields the ALM

$$y_t = (B_0 + B_1)a + (B_0c + B_1c\rho + \varkappa\rho)r^n_{t-1} + \varkappa\epsilon_t. \tag{47}$$

The map from the PLM to the ALM takes the form

$$T(a, c) = ((B_0 + B_1)a, (B_0c + B_1c\rho + \varkappa\rho)). \tag{48}$$

We are now in a position to prove the following proposition.

Proposition 6. *The necessary and sufficient condition for the MSV solution $(0, \bar{c})$ of (43) to be E-stable under interest rate rules with contemporaneous expectations is given by inequality (42).*

Proof. See Appendix F. ■

The Taylor principle condition (42) is also the necessary and sufficient condition for uniqueness of equilibria under contemporaneous expectations. Hence, Proposition 6 shows that the set of parameters consistent with both unique and E-stable equilibria are exactly the same—a conclusion which we also obtained for the specification of contemporaneous data policy rules. The situation for the case of baseline parameter values is again summarized in Figure 1, where the determinate region is also the expectationally stable region.

4. Conclusion

We have studied the stability of a simple macroeconomic system under learning for various monetary policy rules using methods developed by Evans and Honkapohja (1999, 2001). We found that the Taylor principle—that nominal interest rates should be adjusted more than one-for-one with changes in inflation—is closely linked with stability in the learning dynamics across all of our specifications. We also found that, in general, determinacy alone is insufficient to induce learnability of a rational expectations equilibrium. We conclude that it may be unwise to simply assume that coordination on a unique equilibrium can occur under a reasonable description of agent learning.

We stress that the methodology we employ to analyze the effects of learning imparts a lot of information to the agents in our model. By endowing the agents with a perceived law of motion that coincides with the MSV solution of the system, we are in effect giving the agents the correct specification of the vector autoregression they need to estimate in order to learn the rational expectations equilibrium. The local nature of the analysis further imparts initial expectations which are in the immediate neighborhood of the equilibrium. If, under these circumstances, the system is nevertheless driven away from the rational expectations equilibrium, then we do not hold out too much hope that the system can be rendered stable under some other plausible learning mechanism (although of course that remains an open question).[15] For this reason we think of the learnability criterion as a minimal requirement for a policy rule to meet.

In this paper, we have only considered "simple" policy rules, in which policymakers do not respond to the lagged interest rate. In part this was because this is the type of policy

[15] An analogy we have in mind are the notions of *weak* and *strong* E-stability used in the learning literature in univariate models. If a certain equilibrium is not weakly E-stable, then it cannot be strongly E-stable (see Evans and Honkapohja (2001)).

rule studied by Taylor (1993) which fueled the current wave of interest in monetary policy rules. However, estimated policy rules usually include a lagged interest rate in order to better capture the interest rate smoothing observed in actual central bank behavior. In a companion paper, Bullard and Mitra (2000), we are considering the case when the central bank also reacts to a lagged interest rate.

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6. Appendices

A. Proof of Proposition 1

The characteristic polynomial of B (defined by (19)) is given by $p(\lambda) = \lambda^2 + a_1\lambda + a_0$ where

$$a_0 = \frac{\beta\sigma}{\sigma + \varphi_z + \kappa\varphi_\pi}, \tag{49}$$

$$a_1 = \frac{-(\kappa + \sigma + \beta\sigma + \beta\varphi_z)}{\sigma + \varphi_z + \kappa\varphi_\pi}. \tag{50}$$

Both eigenvalues of B are inside the unit circle if and only if both of the following conditions hold (see J.P. LaSalle (1986, p. 28)):

$$|a_0| < 1, \tag{51}$$

$$|a_1| < 1 + a_0. \tag{52}$$

Condition (51) implies the inequality $\varphi_z + \kappa\varphi_\pi > -(1-\beta)\sigma$, which is trivially satisfied since $0 < \beta < 1$. Condition (52), on the other hand, implies (20). The condition, $\kappa(\varphi_\pi - 1) + (1-\beta)\varphi_z \neq 0$, rules out eigenvalues on the unit circle.

B. Proof of Proposition 2

We have spelled out the PLM, the ALM, and the T map from the PLM to the ALM by equations (21), (22), and (23), respectively. Using the results of Evans and Honkapohja (2001), we need the eigenvalues of both B (given by (19)) and ρB to have real parts less than 1 for E-stability. The eigenvalues of ρB are given by the product of the eigenvalues of B and ρ, and since $0 \leq \rho < 1$, it suffices to have only the eigenvalues of B to have real parts less than 1 for E-stability. On the other hand, the MSV solution will not be E-stable if any eigenvalue of B has a real part more than 1. The characteristic polynomial of $B - I$ is given by $\lambda^2 + a_1\lambda + a_2$ where

$$a_1 = \frac{\kappa(2\varphi_\pi - 1) + (2-\beta)\varphi_z + \sigma(1-\beta)}{\sigma + \varphi_z + \kappa\varphi_\pi}, \tag{53}$$

$$a_2 = \frac{\kappa(\varphi_\pi - 1) + (1-\beta)\varphi_z}{\sigma + \varphi_z + \kappa\varphi_\pi}. \tag{54}$$

Both eigenvalues of B have real parts less than 1 (i.e. both eigenvalues of $B - I$ have negative real parts) if and only if $a_1 > 0$ and $a_2 > 0$ (these conditions are obtained by applying the *Routh theorem;* see Chiang (1984)). Note that

$$a_1 = a_2 + \frac{\kappa\varphi_\pi + \varphi_z + \sigma(1-\beta)}{\sigma + \varphi_z + \kappa\varphi_\pi}. \tag{55}$$

So $a_2 > 0$ implies $a_1 > 0$. Consequently, the only condition required is $a_2 > 0$ which reduces to (24). On the other hand, if $\kappa(\varphi_\pi - 1) + (1-\beta)\varphi_z < 0$, then the determinant and trace of $B-I$ are non-zero so that no real root equals zero and in the case of complex eigenvalues, the real parts are non-zero.

C. Proof of Proposition 3

The characteristic polynomial of B (defined by (27)) is

$$p(\lambda) = \lambda^3 - (1+\beta^{-1}+\kappa\beta^{-1}\sigma^{-1})\lambda^2 + (\beta^{-1}-\sigma^{-1}\varphi_z)\lambda + \beta^{-1}\sigma^{-1}(\kappa\varphi_\pi+\varphi_z). \tag{56}$$

Note that $p(0)>0$, $p(-1)<0$ by condition (29), and $p(1)>0$ by condition (28). By Descartes' rule of signs (see Barbeau (1989, p. 171)), there is necessarily a negative root and either two positive roots or a pair of complex conjugates. Since $p(0)>0$ and $p(-1)<0$, the negative root (λ_1) is inside the unit circle. If the roots are all real, then since $\sum_{i=1}^{3}\lambda_i = 1+\beta^{-1}+\kappa\beta^{-1}\sigma^{-1}$ we have

$$\lambda_2+\lambda_3 > 1+\beta^{-1}+\kappa\beta^{-1}\sigma^{-1} > 2+\kappa\beta^{-1}\sigma^{-1}. \tag{57}$$

so that at least one positive root (λ_2) must be more than 1. However, this means that the third positive root also exceeds 1 since $p(1)>0$, $p(\lambda_2+\varepsilon)<0$ for small positive ε, and $p(\infty)=\infty$. If the roots are complex, then the same argument shows their real parts to be more than one.

D. Proof of Proposition 4

The characteristic polynomial of B (given by equation (38)) is $\lambda^2+a_1\lambda+a_0$ where

$$a_0 = \beta(1-\sigma^{-1}\varphi_z), \tag{58}$$

$$a_1 = \kappa\sigma^{-1}(\varphi_\pi-1)+\sigma^{-1}\varphi_z-1-\beta. \tag{59}$$

Both eigenvalues of B are inside the unit circle if and only if conditions (51) and (52) hold. Condition (51) implies inequality (39) whereas condition (52) implies the inequalities (40) and (41).

E. Proof of Proposition 5

The proof of Proposition 2 spells out the expectational stability conditions in this case. We merely note that the characteristic polynomial of $B-I$ is given by

$$\lambda^2 + \{\frac{\kappa(\varphi_\pi-1)+\varphi_z+\sigma(1-\beta)}{\sigma}\}\lambda + \frac{\kappa(\varphi_\pi-1)+(1-\beta)\varphi_z}{\sigma} \tag{60}$$

and it is easy to show that condition (42) is the required condition.

F. Proof of Proposition 6

Appealing to the results of Evans and Honkapohja (2001), for expectational stability, we require the real parts of the eigenvalues of the matrices B_0+B_1 *and* ρB_1+B_0 to have real parts less than one (B_0 and B_1 are defined by (44) and (45)) whereas instability obtains if either matrix has an eigenvalue with real part more than one. Note that B_0+B_1 is identical to the matrix B (given by equation (38)) which was crucial for E-stability under forward expectations. So inequality (42) guarantees that the real parts of the eigenvalues of B_0+B_1 are less than one. The matrix ρB_1+B_0-I has the characteristic polynomial $\lambda^2+a_1\lambda+a_0$ where

$$\begin{aligned} a_1 &= 2-\rho-\beta\rho+\sigma^{-1}\varphi_z-\kappa\sigma^{-1}(\rho-\varphi_\pi), &(61)\\ a_0 &= 1-\rho-\beta\rho+\beta\rho^2+\sigma^{-1}\varphi_z(1-\beta\rho)-\kappa\sigma^{-1}(\rho-\varphi_\pi). &(62)\end{aligned}$$

Note that

$$a_1=a_0+1+\rho(\beta\rho+\beta\sigma^{-1}\varphi_z). \quad (63)$$

so that $a_0>0$ implies $a_1>0$. Consequently, both eigenvalues of (ρB_1+B_0) have real parts less than one if and only if $a_0>0$. We can write a_0 as

$$a_0=\sigma^{-1}[\sigma(1-\rho)(1-\beta\rho)+(1-\beta\rho)\varphi_z+\kappa(\varphi_\pi-\rho)]. \quad (64)$$

The first term within the parentheses is positive. As for the second and third terms, write it as

$$(1-\beta\rho)\varphi_z+\kappa(\varphi_\pi-\rho)=\kappa(\varphi_\pi-1)+(1-\beta)\varphi_z+\kappa(1-\rho)+\beta(1-\rho)\varphi_z. \quad (65)$$

Inequality (42), therefore, implies that $(1-\beta\rho)\varphi_z+\kappa(\varphi_\pi-\rho)>0$, or that, $a_0>0$. Since B_0+B_1 is the same as B (given by (38)), condition (42) is also necessary.

Figure 1
Policy Rules with Contemporaneous Data

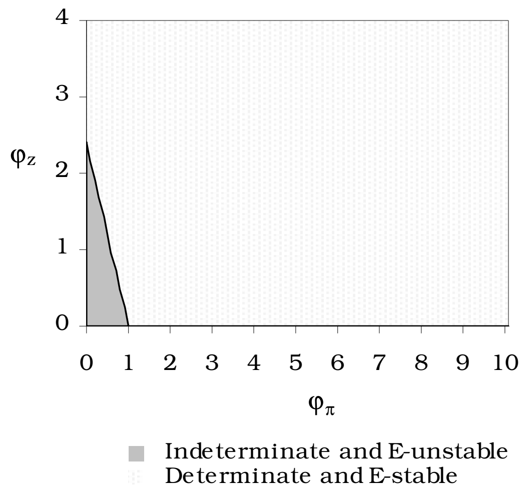

Figure 1: Regions of determinacy and expectational stability for the class of policy rules using contemporaneous data. Parameters other than φ_π and φ_z are set at baseline values.

Figure 2

Policy Rules with Lagged Data

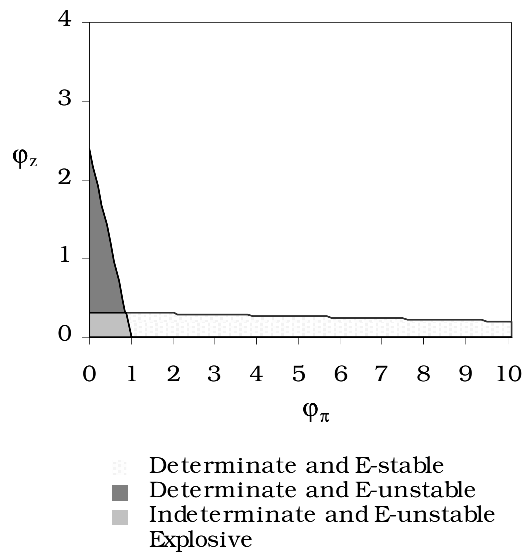

Figure 2: Determinacy and learnability for rules responding to lagged data, with parameters other than φ_π and φ_z set at baseline values. Determinate equilibria may or may not be E-stable.

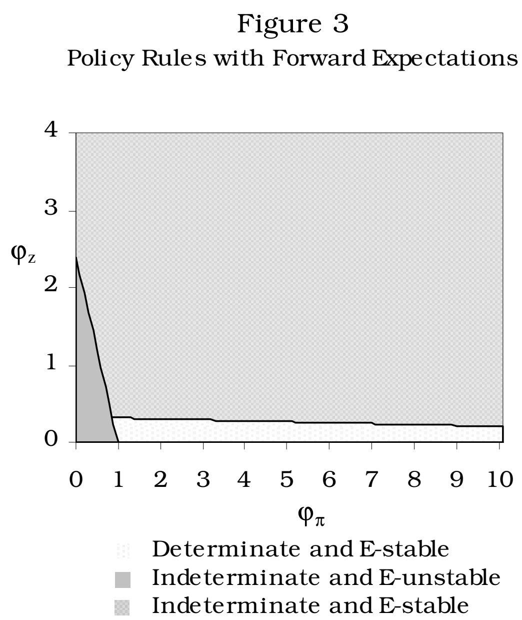

Figure 3: Determinacy and learnability with forward expectations monetary policy rules, at baseline parameter values. Determinate equilibria are always expectationally stable.